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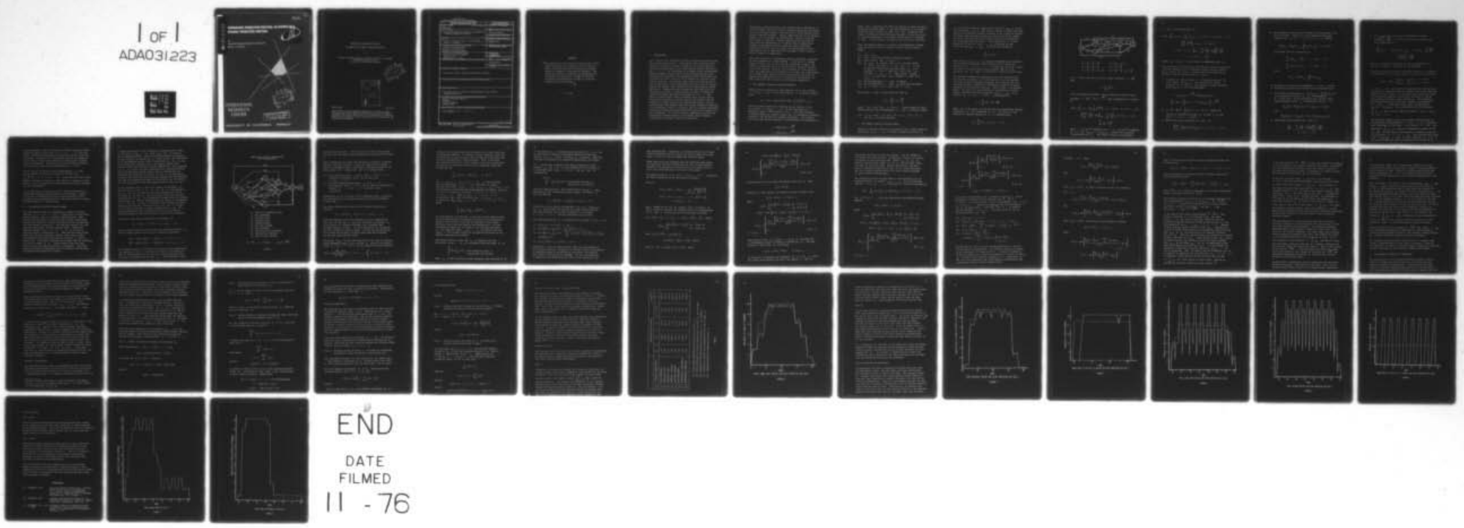

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by
Ronald W. Shephard, Rokaya A. Al-Ayat and
Robert C. Leachman

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SHIPBUILDING PRODUCTION FUNCTION:

AN EXAMPLE OF A DYNAMIC PRODUCTION FUNCTION

by

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ABSTRACT

The structure of the production function for a shipyard is expressed as a dynamic activity analysis model, with the activities related by a directed graph for transfer of intermediate products. The direct and indirect production correspondences are calculated by solution of a system of recursive relations, and application is made to a specific example with load leveling.

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1. INTRODUCTION

Until recently the traditional theory of production functions has been steady state (static). Such models relate exogenous input vectors to net output vectors, except for the static Leontief input-output model which treats intermediate products explicitly. See [1], [2]. The limitations of a steady state correspondence between exogenous inputs and net outputs become apparent when considered for processes like shipbuilding where the greater part of production activity day by day is devoted to producing intermediate products. The constant input rates over some period of time (steady state) cannot meaningfully relate inputs to outputs because the dynamic time histories of the inputs greatly affect the result. Prompted by such problems for the modeling of the construction of ships, a dynamic abstract model of production functions has been developed and reported on in [ ]. This abstract dynamic model relates exogenous input and net output histories by mappings in function spaces with axioms which are extensions of those used for the steady state theory. Having conceptualized the abstract basis for dynamic production functions, it is necessary to turn to a specific form of a dynamic production function for shipbuilding which may serve production planning for this technology. In so doing one must immediately confront the intermediate product outputs of the system, which basically characterize the dynamic aspects of production.

The dynamic production function for shipbuilding is defined on a directed graph, the nodes of which represent production activities, as in an activity analysis, connected by arcs which specify intermediate product transfers. Certain simplifications of the general model in [3] can be made for shipbuilding. Starting with the full model so formulated, further specializations are made to facilitate the development of a computer code to express the dynamic correspondence between input and output histories. Algorithms for this code are given with some examples of the computations for an aggregated (few variables) form of the production graph for shipbuilding.

The graph (network) of intermediate product production activities has the appearance of a PERT NETWORK. In the case of a single output in time, i.e. one unit and not a sequence of units, the production requirements can be generated by a backward calculation on the network. However, for a sequence of output units in time, the model treats the composition of single unit requirements in a dynamic mode, carrying out the interactions over time as against mere superposition of the single unit requirements, so that criticality of an activity is determined dynamically.

2. THE ABSTRACT DYNAMIC PRODUCTION FUNCTION

Denote output histories on a time interval $[0,+\infty)$ by a vector $u = (u_1,u_2, \ldots, u_m)$ of such histories for m different outputs, with

$$u_i : u_i(t) = \text{output/unit time} , \quad \begin{matrix} t \in [0,+\infty) \\ i = 1,2, \ldots, m \end{matrix} .$$

Similarly denote input rate histories on the interval $[0,+\infty)$ by a vector $x = (x_1,x_2, \ldots, x_n)$ of such histories for n exogenous inputs. The vectors x and u are taken as points in product spaces of function spaces BM_+ of nonnegative functions, bounded in the Sup Norm and Lebesgue measurable. Specifically the Direct and Inverse production correspondences are expressed as

$$x \in BM_+^n \rightarrow \mathbb{P}(x) \in 2^{BM_+^m}$$

$$u \in BM_+^m \rightarrow \mathbb{L}(u) \in 2^{BM_+^n} ,$$

where $\mathbb{P}(x)$ represents the subset of vectors of output histories realizable from the vector x of input histories, as an element of the set of all subsets of BM_+^m, and $\mathbb{L}(u)$ represents the subset of vectors of input histories yielding at least the vector u of output histories, as an element of the set of all subsets of BM_+^n. In this fashion the mappings are production functions.

Thus, the abstract model is a function (correspondence) between input and output histories, taken to satisfy certain axioms, namely:

$\mathbb{P}$.1 $\mathbb{P}(0) = \{0\}$.

$\mathbb{P}$.2 $\mathbb{P}(x)$ is totally bounded (relatively compact).

$\mathbb{P}$.3 $\mathbb{P}(\lambda x) \supset \mathbb{P}(x)$ for $\lambda \in [1,+\infty)$.

$\mathbb{P}$.4 If $||x|| > 0$, either $\mathbb{P}(x) = \{0\}$ or there exists $\bar{u} \in \mathbb{P}(\bar{\lambda}x)$ for some $\bar{\lambda} > 0$ and $||\bar{u}|| > 0$. In the second case there exists for each scalar $\theta \in (0,+\infty)$ a scalar $\lambda_\theta \in (0,+\infty)$ such that $(\theta\bar{u}) \in \mathbb{P}(\lambda_\theta \cdot x)$. For each $i \in \{1,2, ..., m\}$ there exists $y \in BM_+^n$, $||y|| > 0$, such that $\mathbb{P}(y)$ contains $u \in BM_+^m$ with $||u_i|| > 0$.

$\mathbb{P}$.5 The correspondence $x \to \mathbb{P}(x)$ is closed.

$\mathbb{P}$.5$^+$ The correspondence $x \to \mathbb{P}(x)$ is upper semi-continuous.

$\mathbb{P}$.6 If $u \in \mathbb{P}(x)$, $(\theta u) \in \mathbb{P}(x)$ for $\theta \in [0,1]$.

The norm of $f \in BM_+^\alpha$ as used above is taken as

$$||f|| = \left[\sum_1^\alpha ||f_i||^2 \right]^{\frac{1}{2}}$$

where $||f_i|| = \text{Sup} \{f(t) \mid t \in [0,+\infty)\}$. These properties imply similar ones for the correspondence $u \to \mathbb{L}(u)$, to which is added

$\mathbb{L}$.0 $\{x \mid x \in \mathbb{L}(u) , y \notin \mathbb{L}(u) \text{ for } y \leq x , ||u|| > 0\} =:$ Eff $\mathbb{L}(u)$ is totally bounded.

3. THE GENERAL ACTIVITY ANALYSIS MODEL

Denote the internal activities of production by a finite number of activities $A_1, A_2, ..., A_k$, to which is added an activity A_o

as a source for exogenous inputs and an activity A_{k+1} as a sink for net outputs. Let $x \in BM_+^n$ denote a vector of input rate time histories of exogenous factors available to the production system; including labor services, machine and facility services, energy and fuels, and materials and equipment installed in production but not produced by the system. A distribution of x to the production activities $A_1, A_2, \ldots, A_k$ is denoted by time histories $x_{01}, x_{02}, \ldots, x_{0k}$. As a physical constraint

$$\sum_{\alpha=1}^{k} x_{0\alpha} \leqq x \ .$$

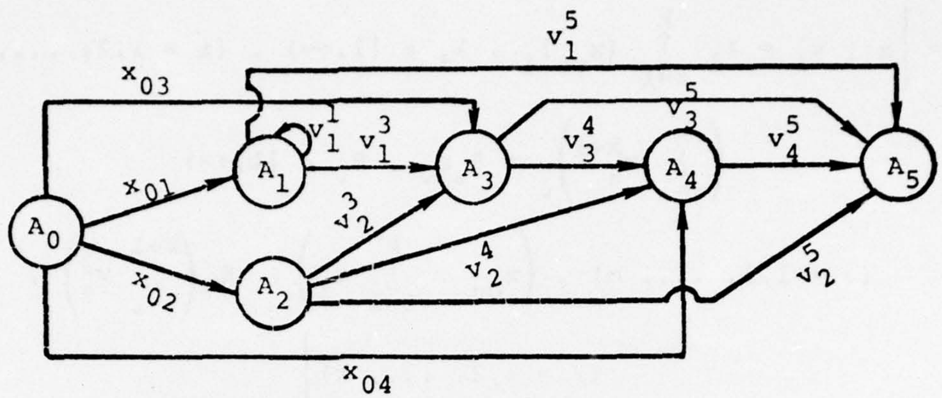
Each activity $A_1, A_2, \ldots, A_k$ produces intermediate and possibly final products (as spares), the time histories of which are denoted by V_α, $\alpha = 1,2, \ldots, k$. Intermediate and final products are taken in a common output function space BM_+^m for time histories, i.e. $V_\alpha \in BM_+^m$, as histories of outputs per unit time.

The possible transfers of intermediate and final products from one activity to another can be displayed by a directed network, the nodes of which represent production activities and the arcs represent possible transfers of intermediate products as inputs exogenous to the receiving activity. Such a graph is illustrated below. The vector of time histories of intermediate and final product transfers from the activities $A_1, A_2, \ldots, A_k$, to an activity A_α $(\alpha = 1,2, \ldots, (k+1))$ is represented by

$$v_\alpha = \sum_{i=1}^{k} v_i^\alpha \ , \ v_i^\alpha \in BM_+^m \ ,$$

where v_i^α is the vector of time histories of intermediate products produced by A_i and transferred to A_α. The vector of time histories of outputs produced by A_α is represented by

$$V_\alpha = \sum_{j=1}^{k+1} v_\alpha^j \ , \ (\alpha = 1,2, \ldots, k) \ .$$



$$V_1 = v_1^1 + v_1^3 + v_1^5 \qquad v_1 = v_1^1 \, , \; v_2 = 0$$

$$V_2 = v_2^3 + v_2^4 + v_2^5 \qquad v_3 = v_1^3 + v_2^3$$

$$V_3 = v_3^4 + v_3^5 \, , \; V_4 = v_4^5 \qquad v_4 = v_2^4 + v_3^4 \, , \; v_5 = v_4^5$$

Let u denote the vector of final output histories, $u \varepsilon BM_+^m$. Then

$$u = \sum_{\alpha=1}^{k} v_\alpha^{k+1} \, .$$

With the foregoing notation, one may express the overall correspondence $x \varepsilon BM_+^n \rightarrow \mathbb{P}(x) \varepsilon 2^{BM_+^m}$ under disposability of outputs by

$$\mathbb{P}(x) = \Bigg\{ u \mid u_\nu = \frac{1}{\theta_\nu} \sum_{\alpha=1}^{k} \left(v_\alpha^{k+1}\right)_\nu \, , \; \theta_\nu \varepsilon [0,1] \, , \; (\nu = 1,2, \ldots, m) \, ,$$

$$\left(\sum_{j=1}^{(k+1)} v_\alpha^j\right) \varepsilon \; \mathbb{P}_\alpha \left(x_{o\alpha} \, , \; \sum_{i=1}^{k} v_i^\alpha\right) , \; (\alpha = 1,2, \ldots, k) \, ,$$

$$\sum_{\alpha=1}^{k} x_{o\alpha} \leqq x \Bigg\} \, ,$$

where u_ν is the ν^{th} component of u, $(\cdot)_\nu$ is the ν^{th} component of $(\cdot)$, and $(x_{o\alpha}, v_\alpha) \rightarrow \mathbb{P}_\alpha(x_{o\alpha}, v_\alpha)$ denotes the production correspondence for A_α. The overall inverse correspondence

$u \to \mathbb{L}(u)$ in the same terms is:

$$\mathbb{L}(u) = \left\{ x \mid x_i = \lambda_i \sum_{\alpha=1}^{k} (x_{0\alpha})_i \ , \ \lambda_i \in [1,+\infty) \ , \ (i = 1,2, \ldots, n) \ , \right.$$

$$\left(\sum_{\alpha=1}^{k} v_\alpha^{k+1} \right)_\nu = \theta_\nu u_\nu \ , \ \theta_\nu \in [1,+\infty)$$

$$(\nu = 1,2, \ldots, m) \ , \ \left(x_{0\alpha} \ , \ \sum_{i=1}^{k} v_i^\alpha \right) \in \mathbb{L}_\alpha \left(\sum_{j=1}^{k+1} v_\alpha^j \right) \ ,$$

$$\left. (\alpha = 1,2, \ldots, k) \right\}$$

where $(V_\alpha) \to \mathbb{L}_\alpha(V_\alpha)$ is the inverse correspondence for A_α .

This expression of the activity analysis model does not explicitly allow for storage of the intermediate product outputs received by an activity for inputs. Allowance for such storage may be handled in the following way for $x \to \mathbb{P}(x)$:

1) Introduce for each activity A_α a follow-on activity $\hat{A}_\alpha$ which inputs $\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha)$ and transforms the output histories of $\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha)$ into cumulative output histories $\hat{\mathbb{P}}_\alpha(\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha))$, by a correspondence

$$\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha) \in 2^{BM_+^m} \to \hat{\mathbb{P}}_\alpha(\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha)) =$$

$$\left\{ \hat{V} \mid \hat{V}(t) = \int_0^t V(\tau)d\tau \ , \ V \in \mathbb{P}_\alpha(x_{0\alpha}, v_\alpha) \right\} \in 2^{BM_+^m} \ .$$

2) Let $\hat{v}_i^\alpha : \hat{v}_i^\alpha(t) = \int_0^t v_i^\alpha(\tau)d\tau$, $t \in [0,+\infty)$, denote the cumulative transfers from node A_i to node A_α at the time t , $\alpha = 1,2, \ldots, (k+1)$.

3) Modify the second constraint of $\mathbb{P}(x)$ to:

$$\left(\sum_{j=1}^{(k+1)} \hat{v}_\alpha^j \right) \in \hat{\mathbb{P}}_\alpha(\mathbb{P}_\alpha(x_{0\alpha}, v_\alpha)) \ , \ (\alpha = 1,2, \ldots, k) \ .$$

4) Most exogenous inputs are not storable, being services of one kind or another. However for those components which are storable, say $\{\nu_1, \nu_2, \ldots, \nu_s\} \subset \{1,2, \ldots, n\}$, define

$$(\hat{x}_{O\alpha})_{\nu_i} : (\hat{x}_{O\alpha}(t))_{\nu_i} = \int_0^t (x_{O\alpha}(\tau))_{\nu_i} d\tau \ , \ t \in [0,+\infty)$$

and express the last constraint as

$$\sum_{\alpha=1}^{k} (\hat{x}_{O\alpha})_{\nu_i} \leqq (\hat{x})_{\nu_i} \qquad i \in \{1,2, \ldots, s\}$$

$$\sum_{\alpha=1}^{k} (x_{O\alpha})_{\nu_i} \leqq (x)_{\nu_i} \qquad i \in \{s+1, \ldots, n\}$$

where

$$(\hat{x})_{\nu_i} : (\hat{x}(t))_{\nu_i} = \int_0^t (x(\tau))_{\nu_i} d\tau \ .$$

In the case of the inverse correspondence $u \to \mathbb{L}(u)$, storage of intermediate products may be handled in the following way:

1) The inverse correspondence of $\mathbb{P}_\alpha(x_{O\alpha}, v_\alpha) \to \hat{\mathbb{P}}_\alpha(\mathbb{P}_\alpha(x_{O\alpha}, v_\alpha))$ at the follow-on activity $\hat{A}_\alpha$ is a correspondence $V_\alpha \to \hat{\mathbb{L}}_\alpha(\mathbb{L}_\alpha(V_\alpha))$ which accumulates over time the intermediate product inputs v_α transferred to $\mathbb{L}_\alpha(V_\alpha)$, i.e.

$$(x_{O\alpha}, \hat{v}_\alpha) \in \hat{\mathbb{L}}_\alpha(\mathbb{L}_\alpha(V_\alpha)) \iff (x_{O\alpha}, v_\alpha) \in \mathbb{L}_\alpha(V_\alpha) \ ,$$

and

$$\hat{\mathbb{L}}_\alpha(\mathbb{L}_\alpha(V_\alpha)) = \left\{(x_{O\alpha}, \hat{v}_\alpha) \mid \hat{v}_\alpha \in \hat{\mathbb{P}}_\alpha(\mathbb{P}_\alpha(x_{O\alpha}, v_\alpha))\right\} .$$

2) Modify the third constraint of $\mathbb{L}(u)$ to

$$\left(x_{O\alpha} \ , \ \sum_{i=1}^{k} \hat{v}_i^\alpha\right) \in \hat{\mathbb{L}}_\alpha\left(\mathbb{L}_\alpha\left(\sum_{j=1}^{k+1} v_\alpha^j\right)\right) ,$$

$$(\alpha = 1,2, \ldots, k) \ .$$

3) If some exogenous inputs may be accumulated, modify $\hat{\mathbb{L}}_\alpha\left(\mathbb{L}_\alpha\left(\sum_{j=1}^{k+1} v_\alpha^j\right)\right)$ to accumulate the same, and express the third constraint as:

$$\left((\hat{x}_{o\alpha})_{\nu_1}, \cdots (\hat{x}_{o\alpha})_{\nu_s}, (x_{o\alpha})_{\nu_{s+1}}, \cdots (x_{o\alpha})_{\nu_n}, \sum_{i=1}^{k} \hat{v}_i^\alpha\right) \varepsilon$$

$$\hat{\mathbb{L}}_\alpha\left(\mathbb{L}_\alpha\left(\sum_{j=1}^{k+1} v_\alpha^j\right)\right).$$

Thus by a bookkeeping operation for the correspondence $\mathbb{P}_\alpha$ and $\mathbb{L}_\alpha$, storage of inputs may be incorporated.

A capacity constraint for each production activity A_α needs to be imposed for each exogenous input for a particular technology:

$$(x_{o\alpha})_j \leqq (\bar{x}_{o\alpha})_j \begin{cases} (x_{o\alpha})_j : (x_{o\alpha}(t))_j, & t \in [0,+\infty) \\ (\bar{x}_{o\alpha})_j : (\bar{x}_{o\alpha}(t))_j, & t \in [0,+\infty), \end{cases}$$

$j \in \{1,2, \ldots, n\}$, stating that at each point of time the rate of applying an exogenous input is bounded, denying infinite time substitution for inputs, i.e. the total input needed to produce a unit cannot be applied in very short intervals of time at high application rates, without limitation. This constraint applies particularly for the input (inverse) correspondence $V_\alpha \to \mathbb{L}_\alpha(V_\alpha)$. In the case of the output correspondence $(x_{o\alpha}, v_\alpha) \to \mathbb{P}_\alpha(x_{o\alpha}, v_\alpha)$ the bounds imposed by the constraint $\sum_{\alpha=1}^{k} x_{o\alpha} \leqq x$ may or may not supercede those of $(x_{o\alpha})_j \leqq (\bar{x}_{o\alpha})_j$, $j \in \{1,2, \ldots, n\}$.

The foregoing general form of the activity analysis model for the production correspondences $x \to \mathbb{P}(x)$, $u \to \mathbb{L}(u)$ exhibits great complexity of structure. In the case of $x \to \mathbb{P}(x)$, the constraint $\sum_{\alpha=1}^{k} x_{o\alpha} \leqq x$ permits a k-dimensional subset of the set BM_+^n, of time history distributions of exogenous inputs to the activities A_α $(\alpha = 1,2, \ldots, k)$. Each distribution yields a set $\mathbb{P}(x_{01}, x_{02}, \ldots, x_{0k})$ of output histories and $\mathbb{P}(x)$ is the union of such sets over all permissable distributions of

exogenous inputs. The time histories of total outputs of intermediate products at the nodes A_α $(\alpha = 1,2, \ldots, k)$ may have similar freedom of distribution if shared by several activities as inputs. Accumulation of intermediate product and exogenous inputs adds a further time substitution possibility of similar order of complexity. Thus one can see the possible complexity of structure for the overall correspondence $x \to \mathbb{P}(x)$.

In the case of the overall input correspondence $u \to \mathbb{L}(u)$, the net output histories are required to satisfy only $\sum_{\alpha=1}^{k} \left(v_\alpha^{k+1}\right)_\nu = u_\nu$ $(\nu = 1,2, \ldots, m)$ in the efficient case, which permits a potential latitude of distribution of sources of final outputs. Again intermediate products may have a complex freedom of time distribution.

It is clear that a calculation in full possible complexity of the general activity analysis model is not possible. However, in the case of shipbuilding certain simplifications of this structure can be reasonably made as a practical matter in making the model specific to shipbuilding.

4. A SHIPBUILDING ACTIVITY ANALYSIS MODEL

For this specific case it is reasonable to assume that each activity A_α $(\alpha = 1,2, \ldots, (k-1))$ yields only a single intermediate product, with A_k yielding completed units. Each activity can be modeled in terms of time variable input-output coefficients driven by a scalar intensity function representing the time history of operation. The resulting system is an open Leontief-like dynamic activity analysis model. Beyond the intrinsic bounds $(\bar{x}_{o\alpha})_j$ $(j = 1,2, \ldots, n)$ on the rate of applying inputs to an activity A_α , there may be certain fixed facilities and major equipment (cranes) which limit further the intensity with which the production activity A_α may be carried out. Such bounds determine the minimum time for completion of a unit of output at A_α . These minimal times set lower bounds to the time lags encountered in production. The actual lags are determined endogenously by the intensity function used in operating the production activity.

A simplified example of the expression of shipbuilding in terms of activities $A_1, A_2, \ldots, A_8$, with $A_{k+1} = A_9$, is given in the accompanying figure. The construction activity for erection of hull structure is denoted by A_1 , which yields a hull structure as an intermediate product input to activities A_2 , A_3 , A_4 , A_5 of installing the propulsion unit, the electric plant, the command control and auxiliary systems respectively. The activities A_2 , A_3 , A_4 yield installed propulsion unit, installed electric plant and installed command control system, respectively, as intermediate products to activities A_6 and A_7 yielding ship equipment and outfitting and furnishings respectively which deliver the same as intermediate products to an activity A_8 of construction servicing which yields the completed unit to the sink A_9 .

At the aggregate level in which the example is expressed, A_1 delivers the same unit to A_2 , A_3 , A_4 and A_5 simultaneously, and A_2 , A_3 , A_4 each deliver the same completed unit simultaneously to A_6 and A_7 , with the latter delivering the same completed unit to A_8 . For bookkeeping purposes with our notation, one may have A_1 deliver simultaneously one fourth of a completed unit to A_2 , A_3 , A_4 and A_5 with the latter delivering simultaneously one half of a completed unit to A_6 and A_7 , which in turn deliver together a single completed unit to A_8 . At a more disaggregated level of modeling, the activities can be arranged perhaps so that partitioning of output is not required.

Returning to the general expression of our model, let

$$Z_\alpha : Z_\alpha(t) \ , \ t \in [0,+\infty) \ , \ \alpha \in \{1,2, \ldots, k\}$$

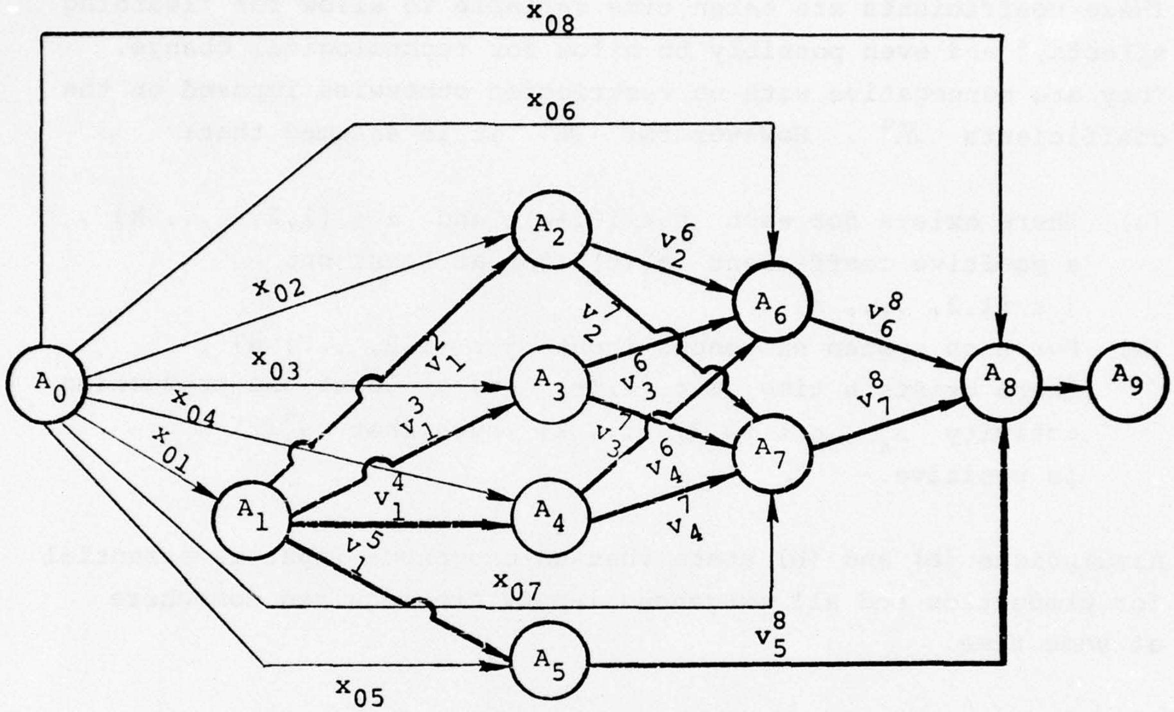
denote a scalar intensity function for the production activity A_α . This intensity function drives input coefficient functions

$$\mathbb{A}^\alpha = ||a_1^\alpha(t), a_2^\alpha(t), \cdots a_n^\alpha(t)|| \ , \ \alpha = 1,2, \ldots, k$$

$$\bar{\mathbb{A}}^2 = ||\bar{a}_1^\alpha(t), \bar{a}_2^\alpha(t), \cdots \bar{a}_{k-1}^\alpha(t)|| \ , \ \alpha = 1,2, \ldots, k$$

for exogenous inputs and intermediate product inputs respectively. The coefficient $a_j^\alpha(t)$ represents at the time t the amount of the j^{th} system exogenous input per unit time required per unit intensity of operating the activity A_α . The coefficient $\bar{a}_i^\alpha(t)$

SIMPLIFIED ACTIVITY NETWORK FOR
SHIP CONSTRUCTION



A_0 SOURCE OF EXOGENOUS INPUTS
A_1 HULL STRUCTURE
A_2 PROPULSION UNIT
A_3 ELECTRIC PLANT
A_4 COMMAND-CONTROL
A_5 AUXILIARY SYSTEMS
A_6 SHIP EQUIPMENT
A_7 OUTFITTING AND FURNISHINGS
A_8 CONSTRUCTION SERVICING
A_9 SINK OF FINAL OUTPUTS

$$V_\alpha \in BM_+ , \; v_\alpha = \left(v_1^\alpha, v_2^\alpha, \ldots, v_{k-1}^\alpha\right) \in BM_+^{k-1}$$

FIGURE 1

represents at the time t the amount of the i^{th} activity output per unit time required per unit intensity of operating the activity A_α.

These coefficients are taken time variable to allow for "learning effects," and even possibly to allow for technological change. They are nonnegative with no restriction otherwise imposed on the coefficients $\bar{\mathbb{A}}^\alpha$. However for $\mathbb{A}^\alpha$ it is assumed that:

(a) There exists for each $t \in [0,+\infty)$ and $\alpha \in \{1,2, \ldots, k\}$, a positive coefficient $a_j^\alpha(t)$ for at least one $j \in \{1,2, \ldots, n\}$.

(b) For each system exogenous input $j \in \{1,2, \ldots, n\}$, there exists a time $t \in [0,+\infty)$ and at least one production activity A_α, $\alpha \in \{1,2, \ldots, k\}$ such that $a_j^\alpha(t)$ is positive.

Assumptions (a) and (b) state that an exogenous input is essential for production and all exogenous inputs are required somewhere at some time.

The output coefficients driven by the intensity function are taken as

$$c^\alpha = ||c^\alpha(t)|| \ , \ c^\alpha(t) > 0 \ , \ \alpha = 1,2, \ldots, k \ .$$

They are understood to be the fraction of a unit completed per unit time, at the time t, per unit intensity of operating A_α, if the output is a sizeable discrete unit, otherwise they represent amount (in bulk). The same convention will be applied for the intermediate product inputs. In this way the expression of the model may be made in simpler terms than that required for handling large discrete units.

The bounds $(\bar{x}_{o\alpha})_j$ on the time histories of the rates of applying system exogenous inputs to the activity A_α imply a time history bound $\bar{z}_\alpha : \bar{z}_\alpha(t)$, $t \in [0,+\infty)$ on the intensity function given by

$$\bar{z}_\alpha(t) = \underset{j}{\mathrm{Min}} \left\{ \frac{(\bar{x}_{o\alpha}(t))_j}{a_j^\alpha(t)} \mid j \in \{1,2, \ldots, n\} \right\} \quad (\alpha = 1,2, \ldots, k)$$

otherwise an exogenous input may be applied at a higher rate than is physically possible. In addition certain exogenous inputs may be bounded as to rate of application, because of fixed facilities and major equipment for a given shipyard. Hence in modeling the production system of a given shipyard there may be additional constraints on the rates at which certain exogenous inputs may be applied, say

$$\sum_{\alpha=1}^{k} (x_{o\alpha}(t))_j \leq (\bar{\bar{x}}_o(t))_j \qquad t \in [0,+\infty)$$

for $j \in \{\nu_1, \nu_2, \ldots, \nu_F\} \subset \{1,2, \ldots, n\}$. In the case where the exogenous inputs $\nu_1, \nu_2, \ldots, \nu_F$ each apply to single distinct activities $\mu_1, \mu_2, \ldots, \mu_F$, $\{\mu_1, \mu_2, \ldots, \mu_F\} \subset \{1,2, \ldots, k\}$, such additional constraints may be merged with the bounds $(\bar{x}_{o\alpha}(t))_j$ to obtain a revised set of bounds $\bar{\bar{Z}}_\alpha : \bar{\bar{Z}}_\alpha(t)$, $t \in [0,+\infty)$, $(\alpha = 1,2, \ldots, k)$. When an input ν_j, $j \in \{1,2, \ldots, F\}$ is shared by two or more activities, say $(\alpha = \sigma_1, \sigma_2, \ldots, \sigma_I)$, an allocation $\left(x_{o\sigma_i}(t)\right)_{\nu_j}$, $(i = 1,2, \ldots, I)$, $I \in \{1,2, \ldots, k\}$, where

$$\sum_{i=1}^{I} \left(x_{o\sigma_i}(t)\right)_{\nu_j} = (\bar{\bar{x}}_o(t))_{\nu_j} ,$$

as a distribution of resources, is only one of many possibilities. The full possibilities in production involve all the alternative bounds on the intensity functions so generable, which permit important factor and time substitution possibilities. By this supplementary imposition of bounds on the intensity functions Z_α, the exogenous inputs $\nu_1, \nu_2, \ldots, \nu_F$ are deleted from the vector x and included as part of the physical structure of the specific shipyard being modeled.

The minimal delays or time lags T_α in production for the activities $A_1, A_2, \ldots, A_k$ are determined from the bounds $\bar{\bar{Z}}_\alpha$ by

$$\int_0^{T_\alpha} \bar{\bar{Z}}(\tau) a^\alpha_{j_\alpha}(\tau) d\tau = \text{total amount of input } j_\alpha \text{ required per unit output,}$$

where j_α is the restricting system exogenous input determining $\bar{\bar{Z}}_\alpha$.

If the activity A_α is operated below the bound $\bar{\bar{Z}}_\alpha$, i.e. it is not limiting, the time lag encountered depends upon the time history $Z_\alpha(t)$, $t \in [0,+\infty)$ at which A_α is operated. Thus the time lags in production are determined endogenously as follows:

If t is the time at which a unit enters production in the activity A_α, and $\Delta_\alpha(t)$ is the increment of time at which it is completed under $Z_\alpha(t)$, $t \in [0,+\infty)$, the lag $\Delta_\alpha(t)$ is determined by

$$\int_t^{t+\Delta_\alpha(t)} a^\alpha_{j_\alpha}(\tau) Z_\alpha(\tau)\, d\tau = \begin{array}{l}\text{total amount of input } j_\alpha \\ \text{required per unit output.}\end{array}$$

One final simplification: Since each activity $A_1, A_2, \ldots, A_{k-1}$ produces a single intermediate product output, the vector v_α will be taken as:

$$v_\alpha = \left(v^\alpha_1, v^\alpha_2, \ldots, v^\alpha_{k-1}\right) \quad (\alpha = 1,2, \ldots, k) .$$

Turning now to the output correspondence $(x_{o\alpha}, v_\alpha) \to \mathbb{P}_\alpha(x_{o\alpha}, v_\alpha)$ for A_α, since fixed time lags are not used, calculations for this correspondence may be made discretely on a time grid $t = 0,1,2, \ldots, N, \ldots$ for some convenient unit of time.

The intensity function Z_α is constrained to satisfy $(\alpha = 1,2, \ldots, k)$

(1) $Z_\alpha(t) \geqq 0$, $t = 0,1,2, \ldots, N, \ldots$

(2) $Z_\alpha(t) a^\alpha_j(t) \leqq (x_{o\alpha}(t))_j$, $\begin{cases} j = 1,2, \ldots, n \\ t = 0,1,2, \ldots, N, \ldots \end{cases}$

(3) $Z_\alpha(t) \bar{a}^\alpha_i(t) \leqq \sum_{\tau=1}^{t} v^\alpha_i(\tau) - \sum_{\tau=0}^{(t-1)} Z_\alpha(\tau) \bar{a}^\alpha_i(\tau)$ $\begin{cases} t = 1,2, \ldots, N, \ldots \\ i = 1,2, \ldots, (k-1) \end{cases}$

$Z_\alpha(0) \bar{a}^\alpha_i(0) \leqq 0$

(4) $Z_\alpha(t) \leqq \bar{\bar{Z}}_\alpha(t)$, $t = 0,1,2, \ldots, N, \ldots$

Conditions (1) and (4) merely reflect lower and upper bounds on the intensity function. Condition (2) reflects a given distribution of exogenous inputs to the production activities, and requires that these allocations be not exceeded. The full possibilities in production are generated by the union of the results for all

such distributions. Constraint (3) allows accumulation of transferred intermediate product inputs and requires that the current input in a unit of time not exceed the stock available.

These constraints are expressed with the convention that inputs are applied at uniform rates during each unit interval and are charged at the beginning of each period for the ensuing unit of time, with output assigned to the end of the period.

The maximal value of $Z_\alpha(t)$ for $t = 0,1,2, \ldots, N, \ldots$ permitted by these constraints may be calculated as follows:

From (2)

$$Z^\alpha(t) \leqq R^\alpha(t) \;,\; R^\alpha(t) := \min_{j \varepsilon S^\alpha(t)} \left\{ \frac{(x_{o\alpha}(t))_j}{a_j^\alpha(t)} \right\}$$

$$S^\alpha(t) := \left\{ j \mid j \;\varepsilon\; \{1,2, \ldots, n\} \;,\; a_j^\alpha(t) > 0 \right\},$$

$$t = 0,1,2, \ldots, N, \ldots$$

Note: Assumption (a) for $\mathbb{A}$ implies $S^\alpha(t)$ not empty. If $a_j^\alpha(t)$ varies in time only by "learning effects," the dependence of S^α upon t is moot, but not for technological change.

Let $\sum^\alpha(t) := \left\{ i \mid i \;\varepsilon\; \{1,2, \ldots, (k-1)\} \;,\; \bar{a}_i^\alpha(t) > 0 \right\}$. Define

$$W^\alpha(0) = \begin{cases} \min\left[R^\alpha(0) \;,\; \bar{\bar{Z}}_\alpha(0)\right] & \text{for } \sum^\alpha(0) = \emptyset \\ 0 & \text{for } \sum^\alpha(0) \neq \emptyset \;. \end{cases}$$

Take $Z_\alpha(0) = W^\alpha(0)$, and from (3)

$$Z_\alpha(1)\bar{a}_i^\alpha(1) \leqq v_i^\alpha(1) - W^\alpha(0) \cdot \bar{a}_i^\alpha(0) \;.$$

Then (2), (3), (4) imply $Z^\alpha(1) \leqq W^\alpha(1)$ where

$$W^{\alpha}(1) = \text{Min}\left[R^{\alpha}(1)\ ,\ \bar{\bar{Z}}_{\alpha}(1)\ ,\ N^{\alpha}(1)\right],$$

$$N^{\alpha}(1) = \begin{cases} \underset{i\varepsilon\sum^{\alpha}(1)}{\text{Min}}\left[\dfrac{\sum\limits_{\tau=0}^{1} v_i^{\alpha}(\tau) - W^{\alpha}(0)\cdot \bar{a}_i^{\alpha}(0)}{\bar{a}_i^{\alpha}(1)}\right], & \sum^{\alpha}(1) \neq \emptyset \\ +\infty, & \sum^{\alpha}(1) = \emptyset . \end{cases}$$

Following the policy of taking the maximal value for Z_{α}, take

$$Z_{\alpha}(1) = W^{\alpha}(1) .$$

Continuing in this fashion, the general solution is found to be:

$$Z_{\alpha}(t) = W^{\alpha}(t)\ ,\ t = 0,1,2,\ \ldots$$

where

$$W^{\alpha}(0) = \begin{cases} \text{Min}\left[R^{\alpha}(0)\ ,\ \bar{\bar{Z}}_{\alpha}(0)\right] & \text{for } \sum^{\alpha}(0) = \emptyset \\ 0 & \text{for } \sum^{\alpha}(0) \neq \emptyset \end{cases}$$

$$W^{\alpha}(t) := \text{Min}\left[R^{\alpha}(t)\ ,\ \bar{\bar{Z}}_{\alpha}(N)\ ,\ N^{\alpha}(t)\right],\ t = 1,2,3,\ \ldots$$

$$N^{\alpha}(t) := \begin{cases} \underset{i\varepsilon\sum^{\alpha}(t)}{\text{Min}}\left[\dfrac{\sum\limits_{\tau=0}^{t} v_i^{\alpha}(\tau) - \sum\limits_{\tau=0}^{t-1} W^{\alpha}(\tau)\bar{a}_i^{\alpha}(\tau)}{\bar{a}_i^{\alpha}(t)}\right], & \sum^{\alpha}(t) \neq \emptyset \\ +\infty, & \sum^{\alpha}(t) = \emptyset . \end{cases}$$

$t = 1,2,3,\ \ldots$

Thus given a vector $v_{\alpha} = \left(v_1^{\alpha}, v_2^{\alpha},\ \ldots,\ v_{k-1}^{\alpha}\right)$ of time histories of intermediate product transfers to the activity A_{α}, the time history of output from A_{α} is given by

$$V_{\alpha}(t) = c^{\alpha}(t)\cdot W^{\alpha}(t) \qquad t = 0,1,2,\ \ldots .$$

In the case of a shipyard, the transfers v_i^{α} $(i = 1,2,\ \ldots,\ (k-1))$ are likely to be positive for only a few indices i, and, in

most cases, positive for only one transfer. See the example at the beginning of this section. In substance a single time history of transfer is made from each node. Nodes A_2 , A_3 , A_4 , A_5 receive a single input, nodes A_6 and A_7 receive three inputs and node A_8 receives three. However this may be, when more than one transfer is made from a node, not in fixed proportions, the total output possibilities for the entire system encompass all such possible distributions.

The foregoing solution has been based on nonaccumulation for all exogenous inputs. If inputs $\nu_1, \nu_2, \ldots, \nu_\alpha$ can be accumulated, then for $j \in \{\nu_1, \nu_2, \ldots, \nu_a\}$ constraint (2) should be expressed as

$$\text{(2a)} \quad \sum_{\tau=0}^{t} Z_\alpha(\tau) a^\alpha_{\nu_i}(\tau) \leqq (\hat{x}_{o\alpha}(t))_{\nu_i} \quad ; \; i = \{1,2, \ldots, a\}$$

for $t = 0,1,2, \ldots$. With this modification the general solution becomes:

$$Z^\alpha(t) = \bar{W}^\alpha(t) \; , \; t = 0,1,2, \ldots$$

where

$$\bar{W}^\alpha(0) = \begin{cases} \text{Min} \left[R^\alpha(0) \; , \; \bar{\bar{Z}}_\alpha(0) \; , \; \bar{R}^\alpha(0) \right] & \text{for} \quad \sum^\alpha(0) = \emptyset \\ 0 & \text{for} \quad \sum^\alpha(0) \neq \emptyset \end{cases}$$

$$\bar{W}^\alpha(t) := \text{Min} \left[R^\alpha(t) \; , \; \bar{R}^\alpha(t) \; , \; \bar{\bar{Z}}_\alpha(t) \; , \; N^\alpha(t) \right] \; , \; t = 1,2, \ldots$$

$$\bar{S}^\alpha(t) := \left\{ i \mid i \in \{1,2, \ldots, a\} \; , \; a^\alpha_{\nu_i}(t) > 0 \right\}$$

$$\bar{R}^\alpha(t) := \begin{cases} \underset{i \in \bar{S}^\alpha(t)}{\text{Min}} \left[\dfrac{(x_{o\alpha}(t))_{\nu_i} - \sum_{\tau=0}^{(t-1)} \bar{W}^\alpha(\tau) a^\alpha_{\nu_i}(\tau)}{a^\alpha_{\nu_i}(t)} \right] , & \bar{S}^\alpha(t) \neq \emptyset \\ + \infty \; , & \bar{S}^\alpha(t) = \emptyset \end{cases}$$

$(t = 1,2, \ldots)$

$$\bar{R}^{\alpha}(0) := \begin{cases} \underset{i \varepsilon \bar{S}^{\alpha}(0)}{\text{Min}} \left[\dfrac{(x_{0\alpha}(0))_{\nu_i}}{a^{\alpha}_{\nu_i}(0)} \right], & \bar{S}^{\alpha}(0) \neq \emptyset \\ +\infty, & \bar{S}^{\alpha}(0) = \emptyset \end{cases}$$

$$N^{\alpha}(t) := \begin{cases} \underset{i \varepsilon \sum^{\alpha}(t)}{\text{Min}} \left[\dfrac{\sum\limits_{\tau=0}^{t} v^{\alpha}_i(\tau) - \sum\limits_{\tau=0}^{t-1} \bar{W}^{\alpha}(\tau)\bar{a}^{\alpha}_i(\tau)}{\bar{a}^{\alpha}_i(t)} \right], & \sum^{\alpha}(t) \neq \emptyset \\ +\infty, & \sum^{\alpha}(t) = \emptyset \end{cases}$$

$t = 1,2, \ldots$

$$\bar{V}_{\alpha}(t) = c^{\alpha}(t)\bar{W}^{\alpha}(t), \quad t = 0,1,2, \ldots$$

The inverse correspondence for a production activity A_{α} may be expressed in cumulative output required, say $\hat{\hat{V}}_{\alpha} : \hat{\hat{V}}_{\alpha}(t)$, $t \varepsilon [0,+\infty)$. A finite horizon T will be considered. Let $\sigma = T - t$ $t = T, T-1, T-2, \ldots$, denote the number of time units preceding T . Let N_{α} be the minimal σ such that $\hat{\hat{V}}(\sigma) = 0$. In terms of the time variable $\sigma = 0,1,2, \ldots$ the constraints on the intensity function $Z_{\alpha}(\sigma)$ are:

(1)' $Z_{\alpha}(\sigma) \geqq 0$; $\sigma = 1,2, \ldots$

(2)' $(x_{0\alpha}(\sigma))_j \geqq a^{\alpha}_j(\sigma)Z_{\alpha}(\sigma)$; $j = 1,2, \ldots, n$, $\sigma = 1,2, \ldots$

(3)' $\hat{v}^{\alpha}_i(\sigma) \geqq \hat{v}^{\alpha}_i(1) - \sum\limits_{\tau=1}^{(\sigma-1)} Z_i(\tau)\bar{a}^{\alpha}_i(\tau)$; $i = 1,2, \ldots, (k-1)$, $\sigma = 2,3, \ldots$

(4)' $Z_{\alpha}(\sigma) \leqq \bar{\bar{Z}}_{\alpha}(\sigma)$; $\sigma = 1,2, \ldots$

(5)' $c^{\alpha}(1)Z_{\alpha}(1) \leqq \hat{\hat{V}}_{\alpha}(0) - \hat{\hat{V}}_{\alpha}(1)$

$$c^{\alpha}(\sigma)Z_{\alpha}(\sigma) + \sum_{\tau=1}^{(\sigma-1)} c^{\alpha}(\tau)Z_{\alpha}(\tau) \leqq \hat{\hat{V}}_{\alpha}(0) - \hat{\hat{V}}_{\alpha}(\sigma)$$

$\sigma = 2,3, \ldots$

Constraints (1)' and (4)' have the same meaning as (1) and (4) for the output correspondence. The intensity function is driven by the constraint (5)' taken with (4)'. A policy of producing as much as possible for current requirements is involved in using (5)', subject to the capacity constraint (4)'. Constraints (2)' and (3)' drive exogenous input requirements noncumulatively and intermediate product transfers cumulatively.

Consider $\sigma = 1$. Then

$$Z_\alpha(1) \leqq \frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(1)}{c^\alpha(1)} , \; Z_\alpha(1) \leqq \bar{\bar{Z}}_\alpha(1)$$

and

$$Z_\alpha(1) \leqq \text{Min}\left[\frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(1)}{c^\alpha(1)} , \; \bar{\bar{Z}}_\alpha(1)\right] =: E^\alpha(1) \; .$$

Take $Z_\alpha(1) = E^\alpha(1)$ in order to produce as much as is possible.

For $\sigma = 2$,

$$Z_\alpha(2) \leqq \frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(2) - c^\alpha(1)E^\alpha(1)}{c^\alpha(2)} , \; Z_\alpha(2) \leqq \bar{\bar{Z}}_\alpha(2) \; ,$$

and

$$Z_\alpha(2) \leqq \text{Min}\left[\frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(2) - c^\alpha(1)E^\alpha(1)}{c^\alpha(2)} , \; \bar{\bar{Z}}_\alpha(2)\right] =: E^\alpha(2) \; .$$

Take $Z_\alpha(2) = E^\alpha(2)$ and continue in this fashion to obtain:

$$Z_\alpha(\sigma) = E^\alpha(\sigma) \; , \; \sigma = 1,2, \ldots$$

where

$$E^\alpha(\sigma) := \text{Min}\left[\frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(\sigma) - \sum_{\tau=1}^{(\sigma-1)} c^\alpha(\tau)E^\alpha(\tau)}{c^\alpha(\sigma)} , \; \bar{\bar{Z}}_\alpha(\sigma)\right]$$

$$\sigma = 2,3, \ldots$$

$$E^\alpha(1) = \text{Min}\left[\frac{\hat{\hat{V}}_\alpha(0) - \hat{\hat{V}}_\alpha(1)}{c^\alpha(1)} , \; \bar{\bar{Z}}_\alpha(1)\right] .$$

Then the noncumulative exogenous inputs required are taken conservatively as

$$(x_{0\alpha}(\sigma))_j = a_j^{\alpha}(\sigma) \cdot E^{\alpha}(\sigma) \ , \ \sigma = 1,2, \ldots \ , \ j = 1,2, \ \ldots, \ n$$

and the cumulative intermediate product transfers required are taken conservatively as:

$$\hat{V}_i^{\alpha}(\sigma) = \hat{V}_i^{\alpha}(1) - \sum_{\tau=1}^{(\sigma-1)} \bar{a}_i^{\alpha}(\tau)E^{\alpha}(\tau) \ , \ \begin{matrix} \sigma = 2,3, \ldots \\ i = 1,2, \ \ldots, \ (k-1) \end{matrix}$$

where $\hat{V}_i^{\alpha}(1)$ is a terminal value equal to the total intermediate product transfers required for $\hat{\hat{V}}_{\alpha}(0)$.

The foregoing recursive formulas for the intensity functions for the correspondences $(x_{0\alpha}, v_{\alpha}) \Rightarrow \mathbb{P}_{\alpha}(x_{0\alpha}, v_{\alpha})$, $\hat{\hat{V}}_{\alpha} \rightarrow \hat{\hat{\mathbb{L}}}_{\alpha}(\hat{\hat{V}}_{\alpha})$, $\alpha = 1,2, \ldots, k$ enable the calculation of the overall correspondences (production functions) $(x_{01}, x_{02}, \ldots, x_{0k}) \rightarrow \mathbb{P}(x_{01}, x_{02}, \ldots, x_{0k})$, $\hat{\hat{V}} \rightarrow \hat{\hat{\mathbb{L}}}(\hat{\hat{V}})$.

In the case of the inverse correspondence $\hat{\hat{V}} \rightarrow \hat{\hat{\mathbb{L}}}(V)$, the intensity function Z_k is calculated first. The nodes $A_1, A_2, \ldots, A_k$ are taken to have been ordered so that all intermediate product transfers needed by the activity A_i are obtainable from $A_1, A_2, \ldots, A_{i-1}$, $i = 1,2, \ldots, k$. Because of this ordering one can proceed to node A_{k-1} , imposing as a cumulative output demand function the cumulative inputs required from A_{k-1} by A_k as determined by the computed intensity function Z_k . From this demand history one computes the required intensity function for Z_{k-1} . Then proceed to node A_{k-2} using as demand function the cumulative sum of the inputs required from A_{k-2} by the nodes A_{k-1} and A_k as determined by the computed intensity functions Z_k , Z_{k-1} . In this fashion the dynamically interacting intensity functions $Z_k, Z_{k-1}, \ldots, Z_1$ can be calculated. Then, one merely uses the constraint (2)' as an equality to determine the time histories $x_{0\alpha}$, $\alpha = 1,2, \ldots, k$ of exogenous inputs required by each activity, and the sum $x = \sum_{\alpha=1}^{k} x_{0\alpha}$ as a time history of total inputs, required to efficiently yield the cumulative output history $\hat{\hat{V}}$.

In the calculation of $\hat{\hat{V}} \to \hat{\mathbb{L}}(\hat{\hat{V}})$ a policy was followed of producing as fast as possible the outputs required at each node. The resulting time histories of intensity levels are likely to vary considerably from constant loading. A smoothing of intensity levels for the nodes can be made in the following way:

An activity intensity level Z_α is *critical* at the time t, if $Z_\alpha(t)$ cannot be decreased without increasing the total span of time of the input history $x = \sum_{\alpha=1}^{k} x_{O\alpha}$. By the policy of calculating the intensity functions Z_α $(\alpha = 1,2, \ldots, k)$, the computed functions $Z_\alpha(t)$ will be at the maximal capacity $\bar{\bar{Z}}_\alpha(t)$ permitted. If Z_α is noncritical at the time t, decrease the bound $\bar{\bar{Z}}_\alpha(t)$ until $Z_\alpha(t)$ becomes critical. By proceeding in this way for all activities at all times a new sequence of intensity functions Z'_α is obtained, which does not extend the total span of time of the input history $x = \sum_{\alpha=1}^{k} x_{O\alpha}$, and yields the smoothest loading possible under such constraint. Further smoothing is possible only by lowering certain intensity capacities further under increase of the total span of time of applying exogenous inputs.

The alterations of the input histories obtained by lowering intensity functions to criticality are *Time Substitutions* of the calculated required input histories. The smoothing technique described above is an example of using *time substitutable input histories* for economic purposes. One cannot apply very irregular loading without encountering extra cost, and although the latter is not calculated explicitly, the motivation is clear. All of the alternative input histories $x = \sum_{\alpha=1}^{k} x_{O\alpha}$ obtained by varying the noncritical intensity functions Z_α, without increasing the total project span, are alternative points in BM_+^n on the isoquant of the map set $\hat{\mathbb{L}}(\hat{\hat{V}})$, in the terminology of production theory. Still other points on the isoquant of the input set $\hat{\mathbb{L}}(\hat{\hat{V}})$ are obtained by decreasing the bounds $\bar{\bar{Z}}_\alpha$ below criticality with extension of the project span.

Accumulation of intermediate product outputs has been permitted, without restriction. This may not be the case. If so, a preassigned output history might not be feasible under restriction

of inventories. Under such curcumstances, feasibility can be obtained by specifying only the total number of vessels needed at a certain point of time. Then with a storage-capacity-modified set of constraints (1)' ··· (5)' a feasible program can be calculated and time substitutable input histories developed for load smoothing.

The calculation of the overall output correspondence $(x_{01}, \ldots, x_{0k}) \to \mathbb{P}(x_{01}, \ldots, x_{0k})$ proceeds by going forward from activity A_1 to A_k in the ordered sequence of nodes using the formulas for the intensity functions Z_α obtained from the constraints (1) ··· (4). Here the calculated intensity levels $Z_\alpha(t)$ are either at their upper bounds $\bar{\bar{Z}}_\alpha(t)$ or at lower levels forced by the input histories $x_{o\alpha}$. If an intensity function Z_α is not critical at the time t , it may be lowered to criticality for load smoothing, without altering the output schedule computed at the first pass. It should be noted here that the output history computed is merely one element of a total set of such output histories obtainable by redistribution of the total input history x to histories $x_{o\alpha}$, subject to $\sum_{\alpha=1}^{k} x_{o\alpha} \leqq x$. Each such output history computable is a point in BM_+^m on the isoquant (production frontier) of the output set $\mathbb{P}(x)$, in the terminology of production theory.

Additional points on the isoquants of $\hat{\mathbb{L}}(\hat{\hat{V}})$ and $\mathbb{P}(x_{01}, \ldots, x_{0k})$ are obtainable by altering the assignments of the services of fixed plant and machinery leading to the intensity capacity bounds $\bar{\bar{Z}}_\alpha$, so that alternative capacity bounds are obtained.

The richness of the production possibilities in a dynamic model is well illustrated here in this case of a shipyard. Thus one can appreciate the need for a dynamic theory of production functions and the limitations of our traditional steady state models.

5. ALGORITHMS AND EXAMPLE OF COMPUTATION

As an application of the foregoing theoretical models, Fortran computer programs have been written for calculating both the direct and inverse correspondences. These programs have been applied to activity networks consisting of up to 22 activities and time horizons as large as 150 time units; it appears that the

programs are limited from handling very large networks and time histories only by available computer storage for variable arrays. The following explains the algorithms used, and the results of applying the programs to the shipyard example of Figure 1 are presented as well.

Much simplification of the calculation of the production correspondences results under the assumption of integral unit transfers of intermediate product. That is, an activity A_α, with immediate predecessor activities* $A_{\beta_1}, \ldots, A_{\beta_m}$, may commence production on output unit n no earlier than epoch τ, where

$$\tau = \mathrm{Min}\left\{t \mid \sum_{\sigma=1}^{t} Z_{\beta_i}(\sigma)c^{\beta_i}(\sigma) \geq n \ , \ i = 1, \ldots, m\right\} .$$

In terms of the shipyard example, activity A_6 (outfitting) cannot begin work on the n^{th} ship until activities A_2, A_3, A_4 (propulsion unit, electric plant, command control) have been completed for the n^{th} ship. Such an assumption is not seriously restrictive. If work on a following activity actually begins after a percentage of a given activity is completed on an output unit, the network may be redefined, splitting the given activity into two "artificial" activities which preserve integral output unit transfers. It suffices to use for each activity a single, scalar output time history; for this purpose, we shall denote the output of activity A_i during $[t-1,t)$ as $u_i(t)$. In the case activity A_i has more than one immediate follower activity, this same time history $u_i(t)$ is made available to each of these follower activities.

Inverse Correspondence

The algorithm for the inverse correspondence uses as input either the desired time history of output units of the final activity or else a cumulative output amount at a certain time horizon (e.g. N ships done by time T) to compute an operations history for the

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*Familiar network terminology is used throughout; for example, activity A_4 is an immediate predecessor of A_6 means there is an arc connecting A_4 and A_6.

activities of minimum overall duration, given the activity intensity bounds and network characteristics. After a modest amount of calculation concerning the criticality of activities, a second iteration yields an operations history for a set of minimal activity intensity peaks (i.e., smoothest activity loading) still maintaining the minimum overall duration and desired final output schedule.

By proceeding backwards through time, the algorithm uses the $u_i(t)$ variables for accumulation purposes as well. Keeping in mind that the output requirements $\{u_i(t)\}$ are what drive the intensity of operation of activity A_i, if $u_i(t)$ exceeds the output capability of the activity, the operation of A_i may be set at capacity and the unfilled output demand added onto the demands of $u_i(t-1)$. The algorithm accomplishes the updating of the outputs of an immediate predecessor activity A_j by merely increasing only outputs $\{u_j(t)\}$ at the latest epochs $\{t\}$ to sufficiently update the accumulated output histories.

Given the order indexing of the activities $A_1, \ldots, A_k$, the algorithm makes use of the corresponding incidence matrix defining the arc network. With this data, the algorithm performs the following three steps on the activities A_i, $i = k, k-1, \ldots, 1$.

Step 1: Compute Intensity of Operation for Activity A_i

Given time horizon T, for $t = T, T-1, \ldots, 1$, set

$$Z_i(t) = \text{Min}\left\{u_i(t)/c^i(t)\ ,\ \bar{\bar{Z}}_i(t)\right\} .$$

If we have set $Z_i(t) = \bar{\bar{Z}}_i(t)$, then set

$$u_i(t-1) := u_i(t-1) + u_i(t) - \bar{\bar{Z}}_i(t)c^i(t)$$

and set

$$u_i(t) := \bar{\bar{Z}}_i(t)c^i(t) .$$

Step 2: Record Epochs Corresponding to Start of Production of Each Output Unit by Activity A_i

Let N be the number of output units to be produced; then for $n = 1, \ldots, N$, define

$$T_i(n) = \mathrm{Min}\left\{\tau \;\middle|\; \sum_{t=1}^{\tau+1} u_i(t) > n - 1\right\}.$$

That is, $T_i(n)$ is the epoch at which activity A_i commences work on output unit n.

Step 3: Update Immediate Predecessor Accumulated Output Histories So As to Provide Necessary Input to A_i

For each immediate predecessor activity A_j of A_i (specified by the incidence matrix), we require

$$\sum_{t=1}^{T_i(n)} u_j(t) \geqq n \;,\; n = 1, \ldots, N \;.$$

In making this test for $n = 1, \ldots, N$, if one encounters for some such n,

$$\sum_{t=1}^{T_i(n)} u_j(t) < n \;,$$

then define

$$W = n - \sum_{t=1}^{T_i(n)} u_j(t)$$

and set

$$u_j(T_i(n)) \; := W + u_j(T_i(n)) \;.$$

It remains to check if part or all of the required additional output (W) is produced at later times; if so, reduce output at these times accordingly. Symbolically,

for $t = T_i(n) + 1, \ldots, T$, set simultaneously

$$W \; := \mathrm{Max}\;\{0, W - u_j(t)\}$$

$$u_j(t) \; := \mathrm{Max}\;\{0, u_j(t) - W\} \;.$$

Having computed the intensity of operation for each activity, the schedule of inputs may then routinely be calculated. Specifically, for each activity α , input type j ,

$$(x_{o\alpha}(t))_j = Z_\alpha(t) a_j^\alpha(t) \ , \ t = 1, \ldots, T \ .$$

Direct Correspondence

The algorithm for the direct correspondence uses as input the time histories of exogenous inputs $x_{o\alpha}(t)$ specified up to a certain horizon T , and the desired accumulated output at T (e.g., the number of ships to be completed). The initial iteration produces an operating schedule of minimum duration, given the activity network characteristics and intensity bounds, and available inputs. As before, a second iteration with the activity intensity bounds revised as to criticality yields the operations history with smooth activity loadings which still maintain the minimum overall production duration.

Given the order indexing of activities $A_1, \ldots, A_k$, the algorithm makes use of the corresponding incidence matrix defining the arc network. With this data the algorithm performs the following three steps on the activities A_i , $i = 1, \ldots, k$.

Step 1: Determine Epochs at Which A_i is Allowed by Predecessor Activities to Begin Production on Each Output Unit

The algorithm determines, for each output unit, the latest epoch τ some immediate predecessor of A_i finishes that output unit; A_i would then be allowed to begin on that output unit at epoch τ .

For each immediate predecessor A_j of A_i (specified by the incidence matrix), for $n = 1, \ldots, N$, set

$$TF_j(n) = \text{Min} \left\{ \tau \mid \sum_{t=1}^{\tau} u_j(t) \geq n \right\}$$

and set

$$TS_i(n) = \text{Max} \ \{TF_j(n) \mid A_j \text{ is an immediate predecessor of } A_i\} \ .$$

For convenience, set

$$TS_1(n) = 0 \ , \ n = 1, \ldots, N$$

and set

$$TS_i(N + 1) = T + 1 \ , \ i = 1, \ldots, k \ .$$

Step 2: Compute Intensity of Operation for Activity A_i Allowed By Exogenous Inputs and Initial Starting Epoch

For $t = 1, \ldots, TS_i(1)$, set $u_i(t) = 0 \quad (i \neq 1)$.
For $t = TS_i(1) + 1, \ldots, T$, set

$$Z_i(t) = \mathrm{Min} \left\{ \bar{\bar{Z}}_i(t) \ , \ \mathrm{Min}_k \ \frac{(x_{oi}(t))_k}{c_k^i(t)} \right\}$$

and set

$$u_i(t) = Z_i(t) c^i(t) \ .$$

Step 3: Revise Intensity and Ouput of A_i to Reflect Unit Starting Epochs Calculated in Step 1

For each output unit $n = 1, \ldots, N$, the algorithm reduces outputs of A_i so that the cumulative output of A_i does not exceed n before epoch $TS_i(n + 1)$. Symbolically, for $\tau = TS_j(n) + 1, \ldots, TS_j(n + 1)$, the algorithm checks the cumulative output, and if for some such τ

$$\sum_{t=1}^{\tau} u_i(t) \geq n \ ,$$

then set

$$u_i(\tau) : = n - \sum_{t=1}^{\tau-1} u_i(t)$$

and set

$$u_i(t) = 0 \ , \ t = \tau + 1, \ldots, TS_j(n + 1)$$

and set

$$Z_i(t) = u_i(t)/c^i(t) \ , \ t = \tau, \ldots, TS_j(n + 1) \ .$$

Example of Computations: Shipbuilding Case

The computer programs, outlined above, were applied to data for the aggregated network of shipbuilding shown in Figure 1. Although the programs in general will accept time-dependent coefficients and time-dependent intensity bounds to accommodate learning effects, in the following computation input and output coefficients $a_j^\alpha(t)$ and $c^\alpha(t)$ were taken to be constant through time. Table 1 shows these coefficients for all nodes of the activity network under consideration.

In this example, labor is taken to be the limiting input determining the operating capacity for each activity. Thus the intensity bounds represent estimates of the maximal labor input, measured in man-weeks per time unit (week), physically possible for each activity. In general, however, any of the inputs to an activity could represent the limiting resource determining an activity's capacity. Moreover, various activities need not have their intensities measured in terms of the available input of the same resource or facility service. Table 1 also shows the capacity bounds $\bar{\bar{z}}^\alpha(t)$, $\alpha = 1, \ldots, 8$.

Backward Program

With the above mentioned data, the inverse (backward) correspondence program was used to compute an operating schedule for 8 ships to be produced within a specified planning horizon of 80 weeks for two final output schedule requirements.

Case I:

A deadline of 80 weeks to complete the 8 ships was specified. An initial program run yielded a schedule of minimum duration beginning at week 41. The solid curve of Figure 2 shows the total labor input per week to the overall network $\left(\sum_{\alpha=1}^{8} x_{o\alpha}(t)\right)$ needed for project completion. The uneven loading apparent in this schedule necessitates a second calculation to achieve a smoother schedule without extending the project duration. For this purpose, a set of minimal constant activity intensity bounds was computed from the first run and used in a second run. Total labor input per week to the shipyard for the smoothed schedule produced in the

		PRODUCTION COEFFICIENTS				CAPACITY
A_α	ACTIVITY DESCRIPTION	a_1^α	a_2^α	a_3^α	$c^\alpha \times 10^{-4}$	$\bar{\bar{z}}_\alpha$
A_1	HULL STRUCTURE	1.00	208.0	106.0	0.93	3165.0
A_2	PROPULSION UNIT	1.00	2750.0	202.0	3.50	840.0
A_3	ELECTRIC PLANT	1.00	1250.0	145.0	5.00	590.0
A_4	COMMAND & CONTROL	1.00	3040.0	213.0	1.82	1620.0
A_5	AUXILIARY SYSTEMS	1.00	525.0	118.0	1.15	2555.0
A_6	SHIP EQUIPMENT	1.00	727.0	125.0	2.20	1335.0
A_7	OUTFITTING & FURNISHING	1.00	1360.0	149.0	7.83	375.0
A_8	CONSTRUCTION SERVICES	1.00	50.0	100.0	1.95	1520.0

a_1^α = labor weeks required per man-week for activity A_α

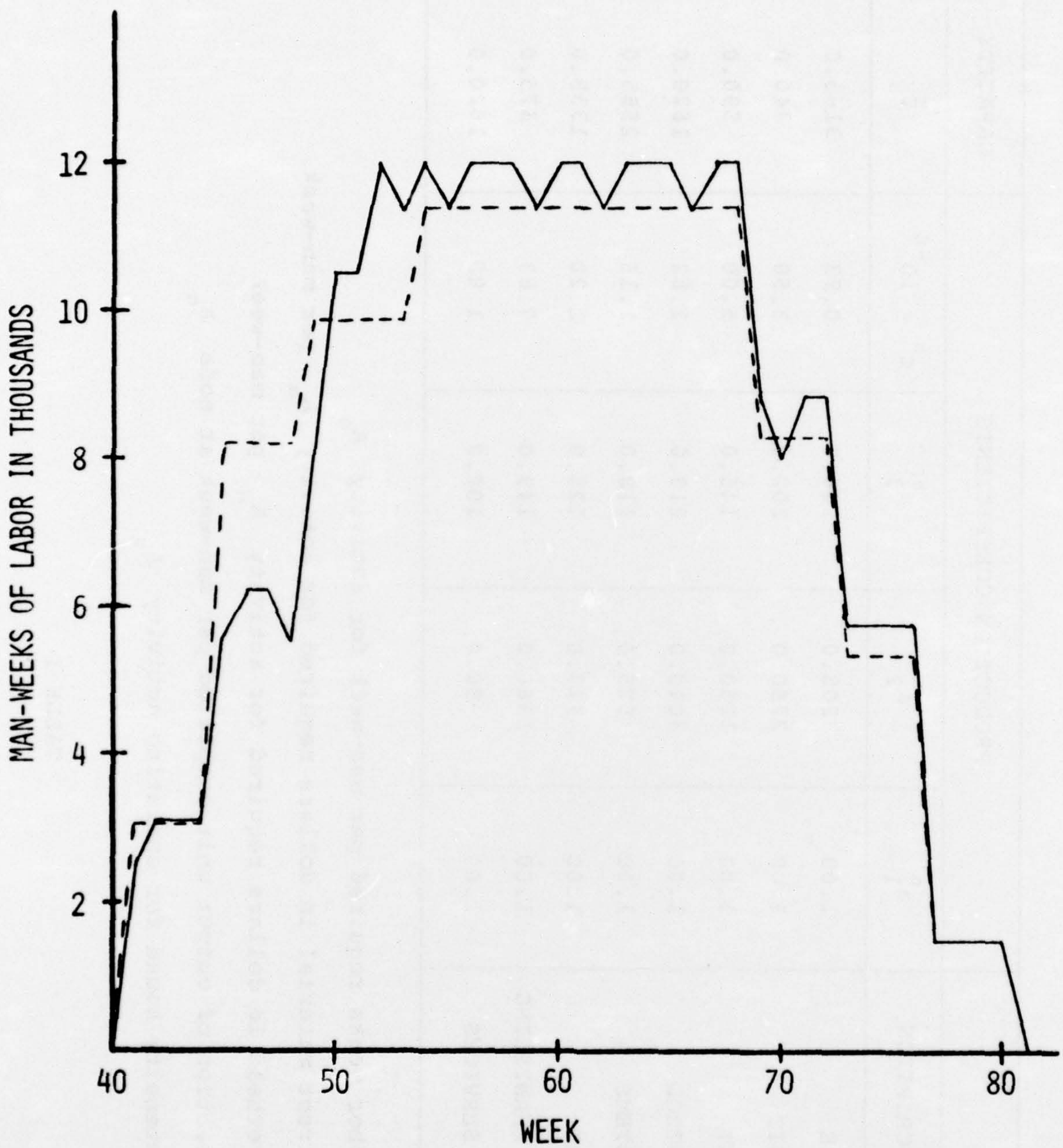
a_2^α = direct material in dollars required for activity A_α per man-week

a_3^α = overhead in dollars required for activity A_α per man-week

c^α = fraction of output unit completed per man-week at node A_α

$\bar{\bar{z}}_\alpha$ = intensity bound for operating activity A_α

TABLE 1



TOTAL LABOR INPUT BEFORE AND AFTER SMOOTHING FOR CASE 1

FIGURE 2

second calculation is shown as the dashed curve of Figure 2. It is to be noted here that the areas under the solid and dashed curves are identical. Figure 3 shows the total material input (dollar value) per week required for the overall network, for each of the two runs, while Figure 4 shows the man weeks of labor per week used by activity A_5 (ship equipment) for each of the two runs.

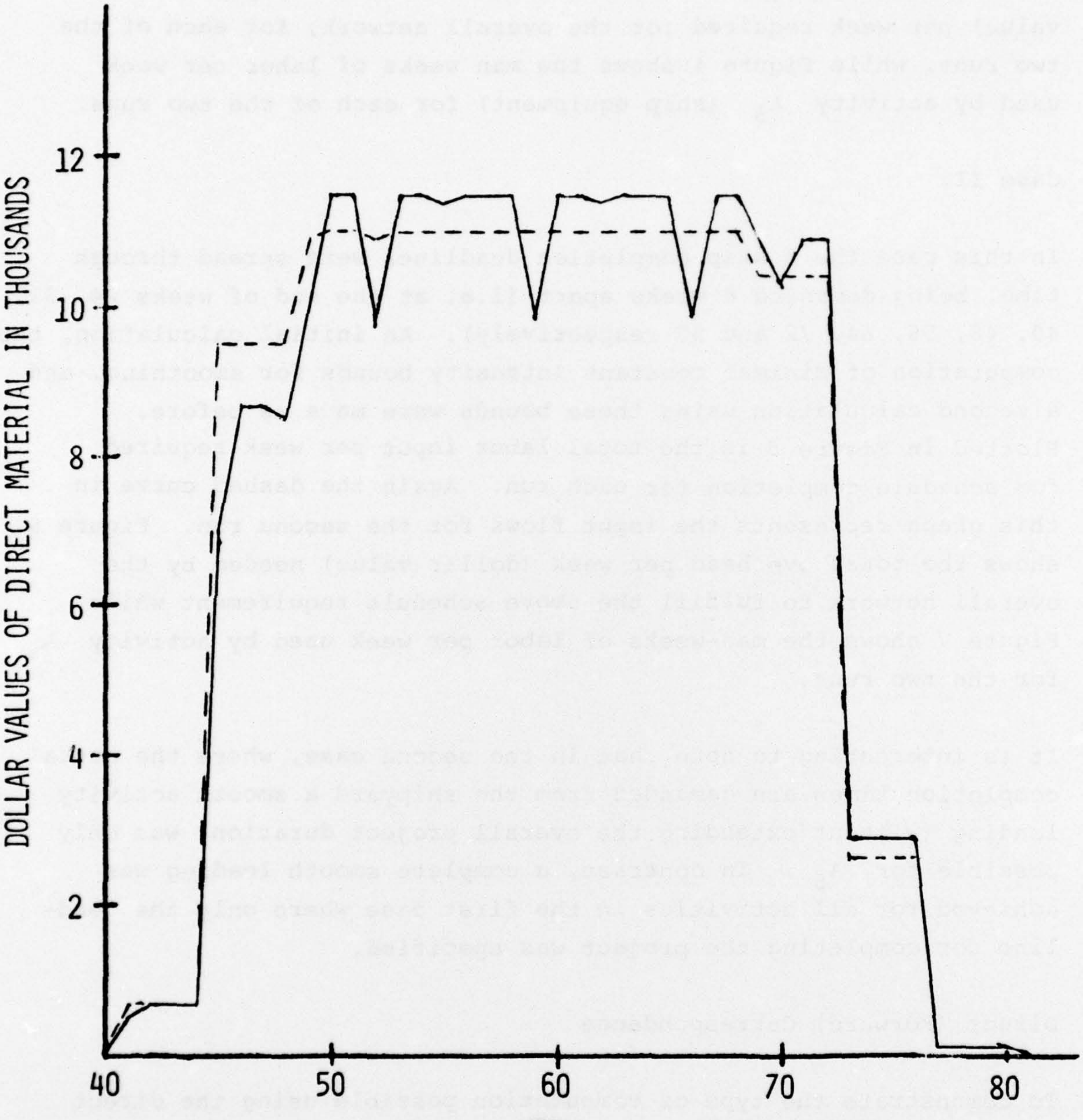
Case II:

In this case the 8 ship completion deadlines were spread through time, being demanded 8 weeks apart (i.e. at the end of weeks 24, 32, 40, 48, 56, 64, 72 and 80 respectively). An initial calculation, the computation of minimal constant intensity bounds for smoothing, and a second calculation using these bounds were made as before. Plotted in Figure 5 is the total labor input per week required for schedule completion for each run. Again the dashed curve in this graph represents the input flows for the second run. Figure 6 shows the total overhead per week (dollar value) needed by the overall network to fulfill the above schedule requirement while Figure 7 shows the man-weeks of labor per week used by activity A_5 for the two runs.

It is interesting to note that in the second case, where the actual completion times are demanded from the shipyard a smooth activity loading (without extending the overall project duration) was only possible for A_5 . In contrast, a complete smooth loading was achieved for all activities in the first case where only the deadline for completing the project was specified.

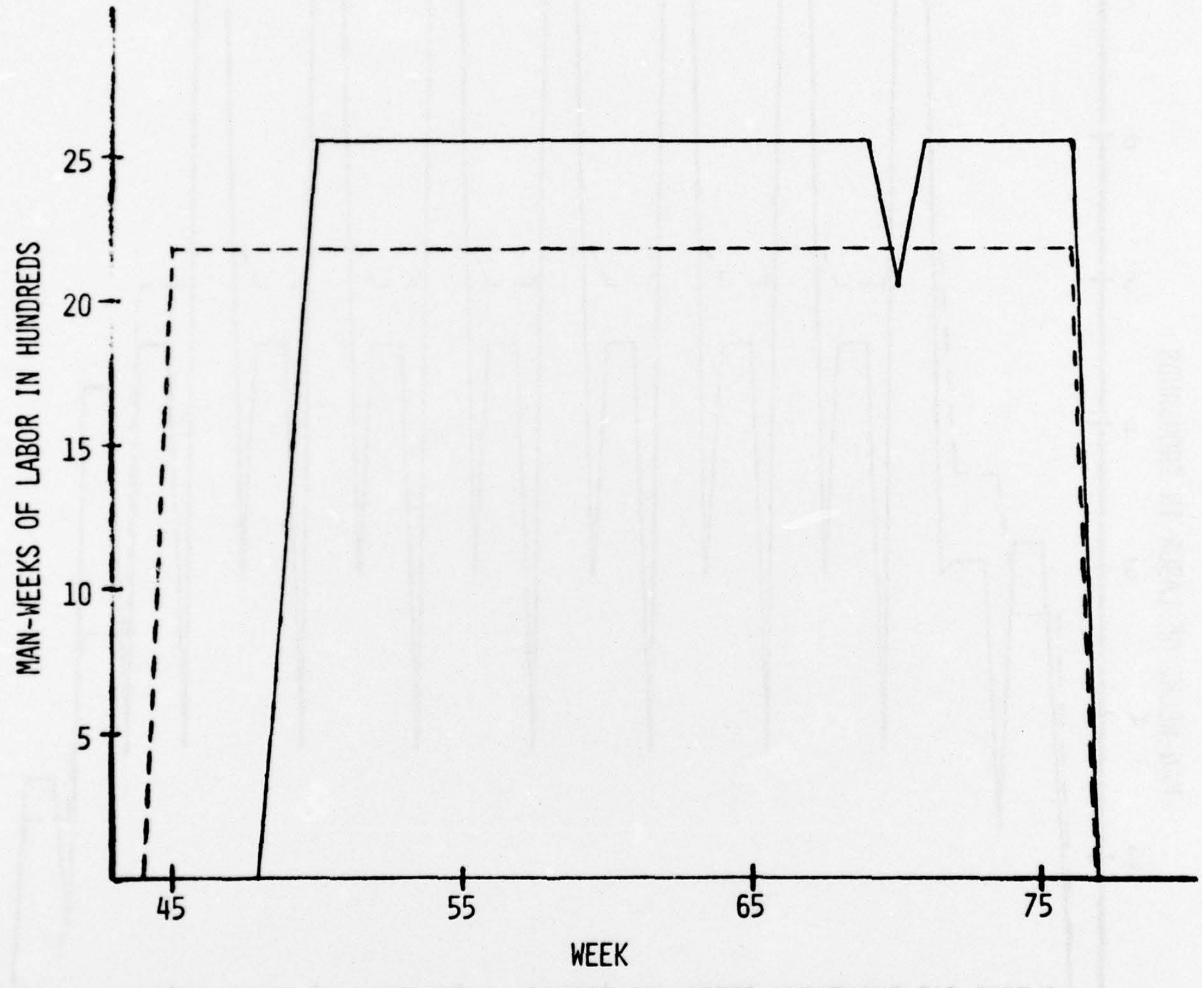
Direct (Forward) Correspondence

To demonstrate the type of computation possible using the direct correspondence program, the same input and output coefficients for the network given in Table 1 were used. The intensity bounds $\bar{\bar{Z}}^{\alpha}$, however, were taken to be those smoothing bounds developed from the first calculation in each of the above two cases for the inverse correspondence. Outputs of the backward program runs were also used as a guideline in determining a set of feasible exogenous input flows for each activity. Specifically, the maximum input flow for each input type for each activity obtained in the smoothed backward calculation was used as a constant input rate available



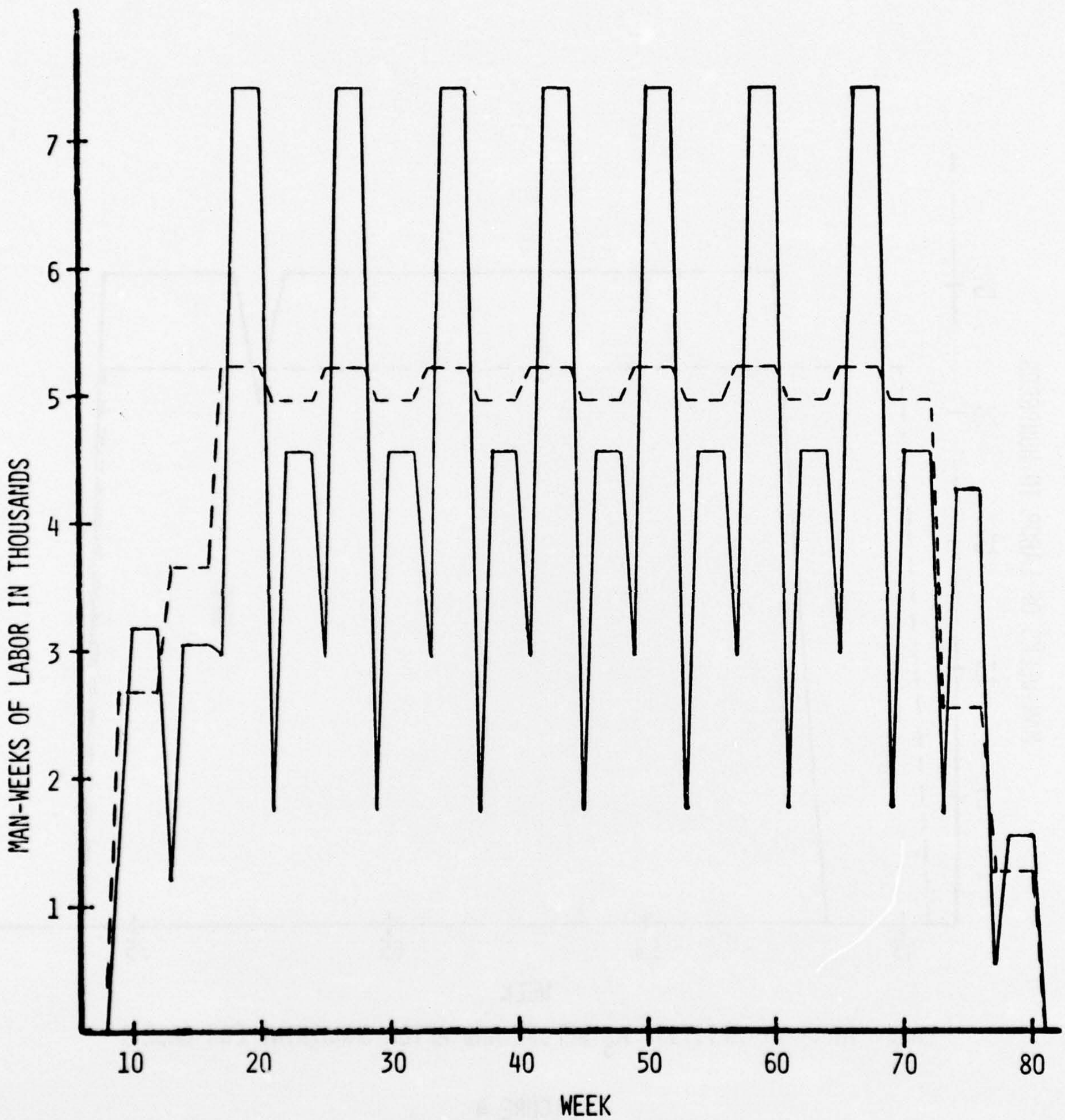
TOTAL MATERIAL BEFORE AND AFTER SMOOTHING FOR CASE 1

FIGURE 3



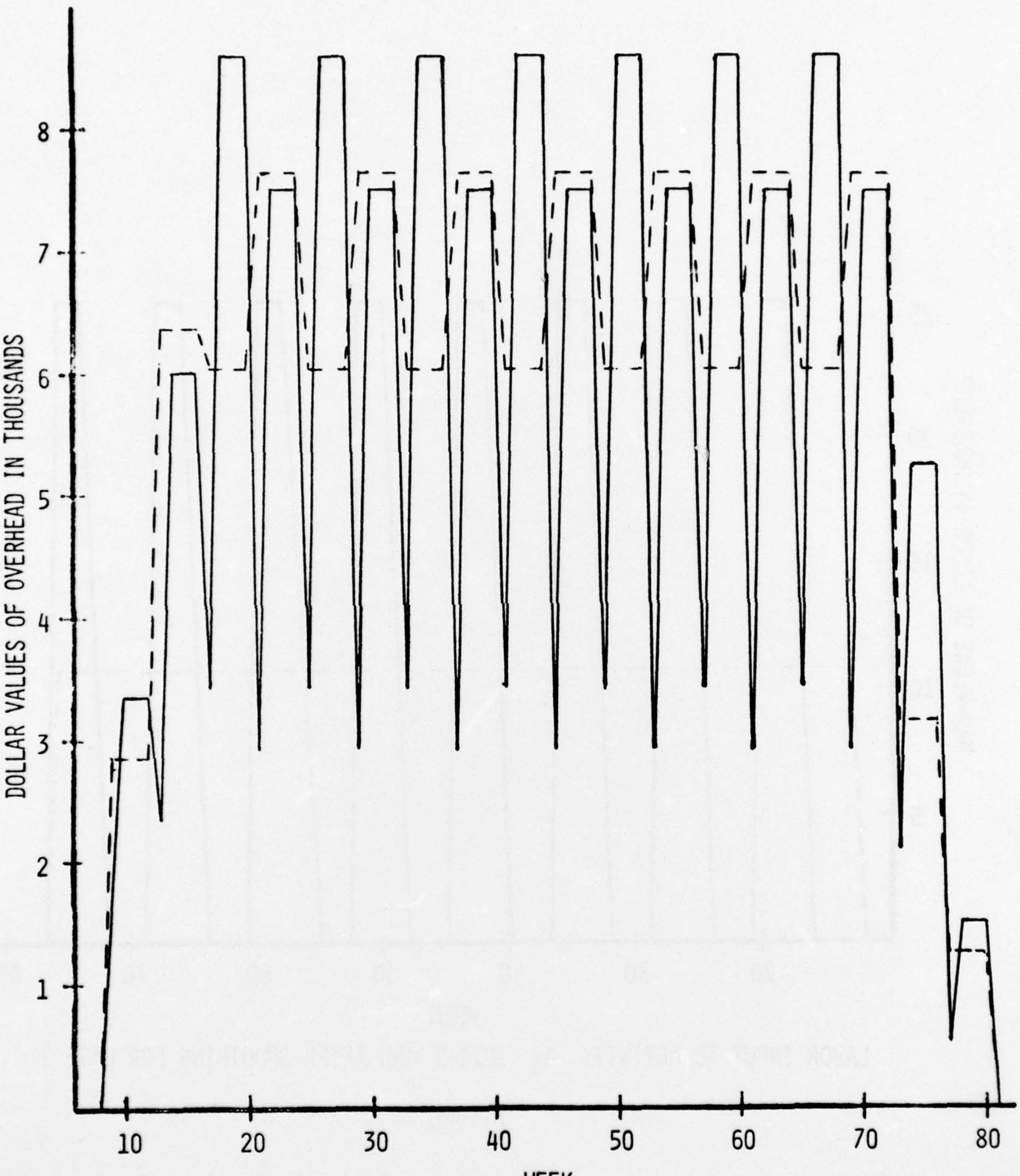
LABOR INPUT TO ACTIVITY A_5 BEFORE AND AFTER SMOOTHING FOR CASE 1

FIGURE 4



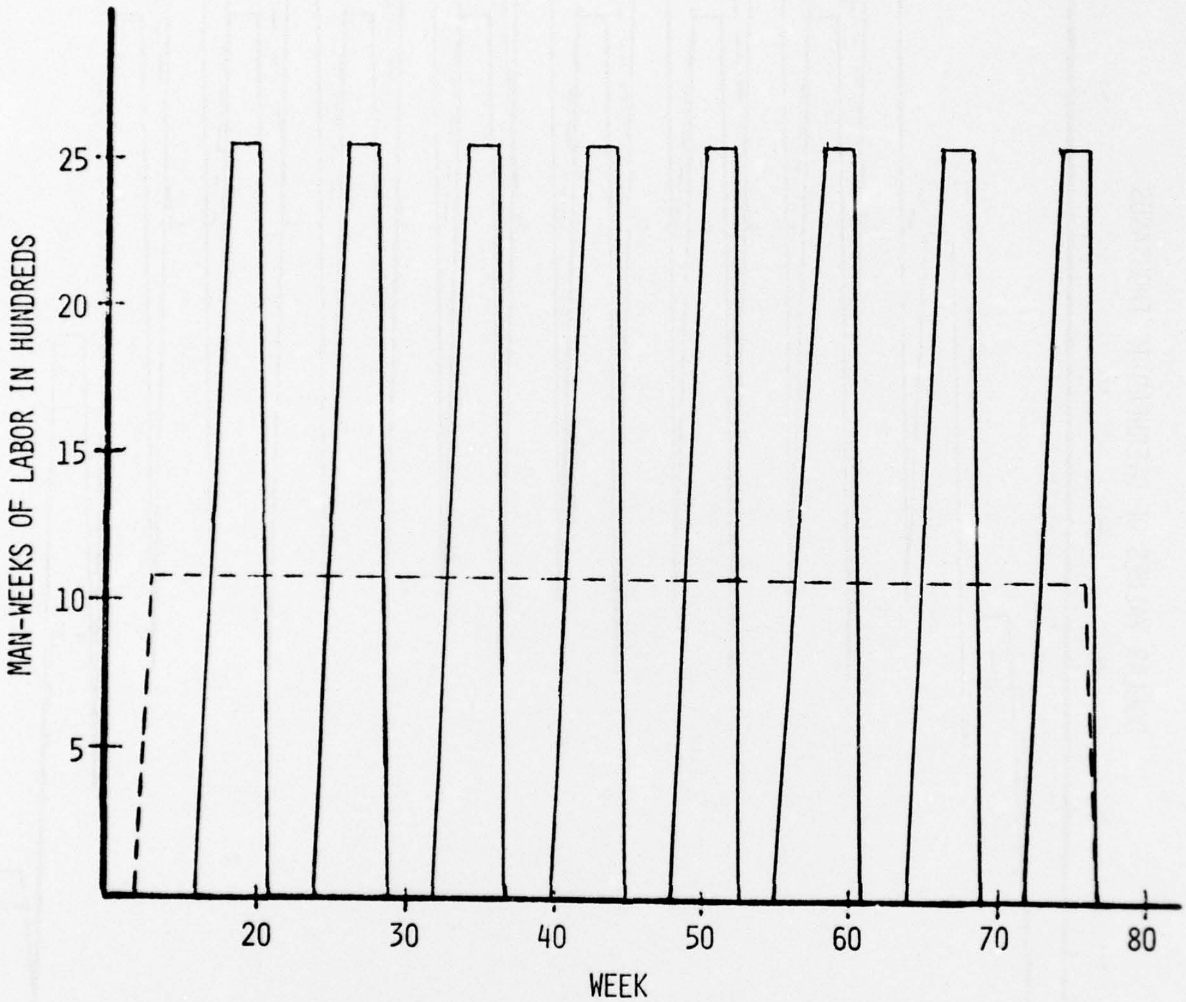
TOTAL LABOR INPUT BEFORE AND AFTER SMOOTHING FOR CASE 2

FIGURE 5



TOTAL OVERHEAD BEFORE AND AFTER SMOOTHING FOR CASE 2

FIGURE 6



LABOR INPUT TO ACTIVITY A_5 BEFORE AND AFTER SMOOTHING FOR CASE 2

FIGURE 7

to the activity.

Case I Data:

The time histories generated by the forward program were, except for a translation of starting times, identical to those generated by the backward program. This is to be expected, since the intensity bounds and the available inputs insured that all activities had smooth loading and no slack time.

Case II Data:

Although the overall project duration and total inputs used were identical to those generated by the backward program, the time histories of input allocation were quite different from their counterparts for the backward calculation. Plotted in Figures 8 and 9 are the total man-weeks per week used by the shipyard necessary for project completion and the total materials used per week (in dollar value) respectively.

Only a few results have been graphed here but these programs generate complete time histories of inputs and outputs. Small aberration in the time interconnections of activities may be caused by the discrete time approach, but such problems diminish as the time increment is reduced.

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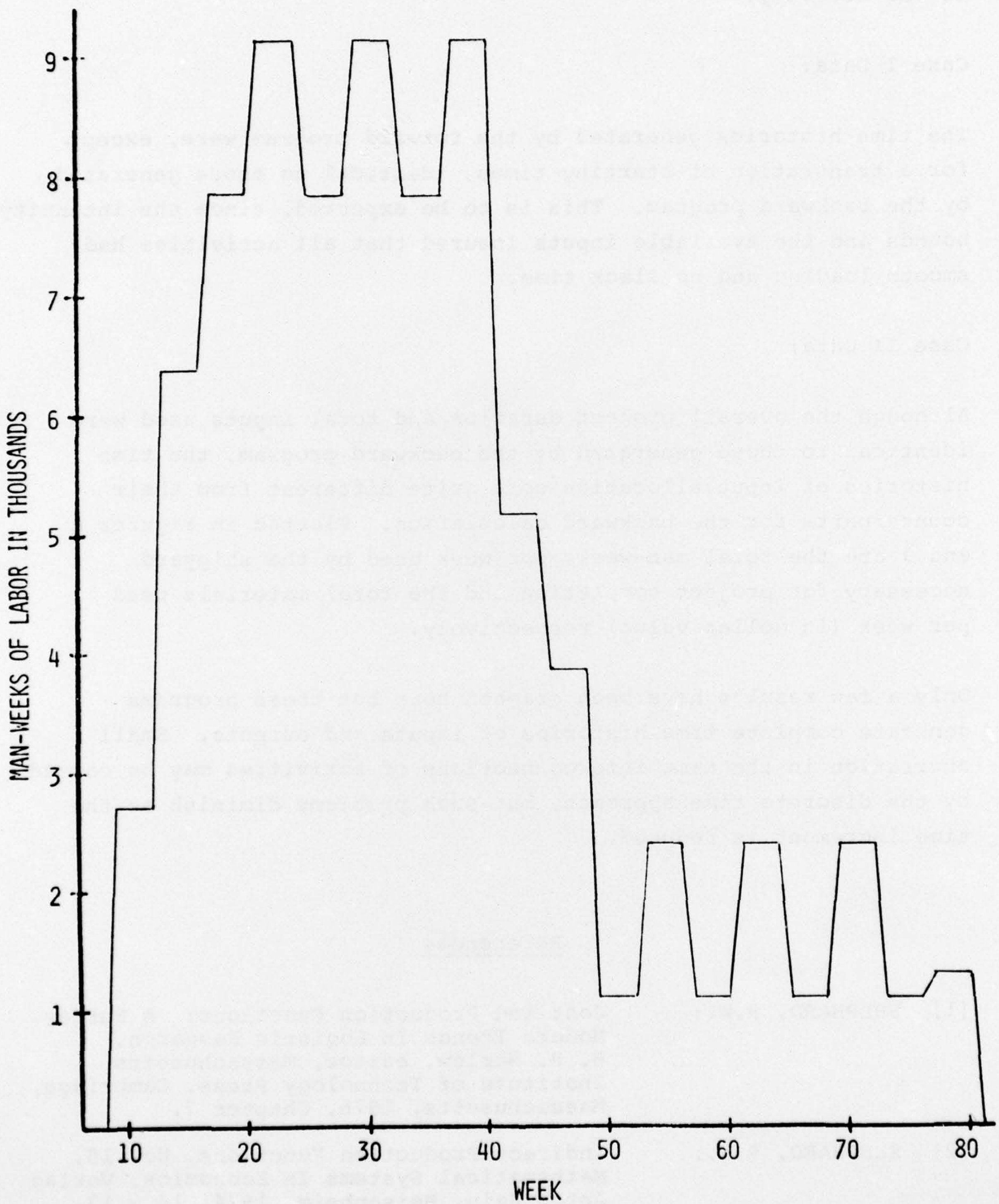
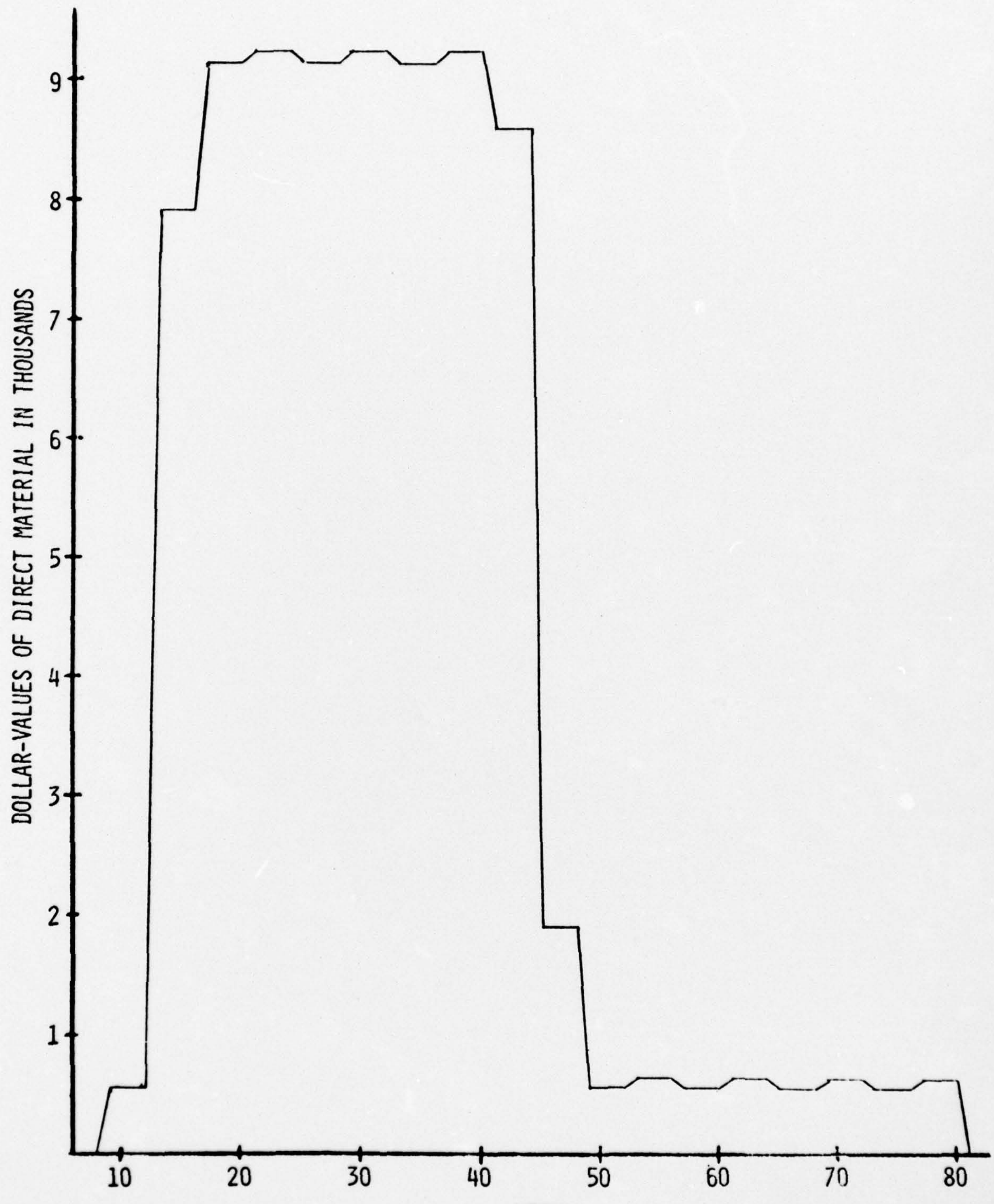


TOTAL LABOR INPUT FOR CASE 2

FIGURE 8



TOTAL INPUT OF MATERIAL FOR CASE 2

FIGURE 9